CFA Institute®
CFA Program

# QUANTITATIVE METHODS

CFA® Program Curriculum
**2023 • LEVEL 1 • VOLUME 1**

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# Quantitative Methods

LEARNING MODULE

# 1

## The Time Value of Money

**by Richard A. DeFusco, PhD, CFA, Dennis W. McLeavey, DBA, CFA, Jerald E. Pinto, PhD, CFA, and David E. Runkle, PhD, CFA.**

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### LEARNING OUTCOME

| *Mastery* | *The candidate should be able to:* |
|---|---|
| ☐ | interpret interest rates as required rates of return, discount rates, or opportunity costs |
| ☐ | explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk |
| ☐ | calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows |
| ☐ | demonstrate the use of a time line in modeling and solving time value of money problems |
| ☐ | calculate the solution for time value of money problems with different frequencies of compounding |
| ☐ | calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding |

## INTRODUCTION

**1**

As individuals, we often face decisions that involve saving money for a future use, or borrowing money for current consumption. We then need to determine the amount we need to invest, if we are saving, or the cost of borrowing, if we are shopping for a loan. As investment analysts, much of our work also involves evaluating transactions with present and future cash flows. When we place a value on any security, for example, we are attempting to determine the worth of a stream of future cash flows. To carry out all the above tasks accurately, we must understand the mathematics of time value of money problems. Money has time value in that individuals value a given amount of money more highly the earlier it is received. Therefore, a smaller amount of money now may be equivalent in value to a larger amount received at a future date. The **time value of money** as a topic in investment mathematics deals with equivalence relationships between cash flows with different dates. Mastery of time value of money concepts and techniques is essential for investment analysts.

The reading[1] is organized as follows: Section 2 introduces some terminology used throughout the reading and supplies some economic intuition for the variables we will discuss. Section 3 tackles the problem of determining the worth at a future point in time of an amount invested today. Section 4 addresses the future worth of a series of cash flows. These two sections provide the tools for calculating the equivalent value at a future date of a single cash flow or series of cash flows. Sections 5 and 6 discuss the equivalent value today of a single future cash flow and a series of future cash flows, respectively. In Section 7, we explore how to determine other quantities of interest in time value of money problems.

## 2 INTEREST RATES

- ☐ interpret interest rates as required rates of return, discount rates, or opportunity costs
- ☐ explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

In this reading, we will continually refer to interest rates. In some cases, we assume a particular value for the interest rate; in other cases, the interest rate will be the unknown quantity we seek to determine. Before turning to the mechanics of time value of money problems, we must illustrate the underlying economic concepts. In this section, we briefly explain the meaning and interpretation of interest rates.

Time value of money concerns equivalence relationships between cash flows occurring on different dates. The idea of equivalence relationships is relatively simple. Consider the following exchange: You pay $10,000 today and in return receive $9,500 today. Would you accept this arrangement? Not likely. But what if you received the $9,500 today and paid the $10,000 one year from now? Can these amounts be considered equivalent? Possibly, because a payment of $10,000 a year from now would probably be worth less to you than a payment of $10,000 today. It would be fair, therefore, to **discount** the $10,000 received in one year; that is, to cut its value based on how much time passes before the money is paid. An **interest rate**, denoted $r$, is a rate of return that reflects the relationship between differently dated cash flows. If $9,500 today and $10,000 in one year are equivalent in value, then $10,000 – $9,500 = $500 is the required compensation for receiving $10,000 in one year rather than now. The interest rate—the required compensation stated as a rate of return—is $500/$9,500 = 0.0526 or 5.26 percent.

Interest rates can be thought of in three ways. First, they can be considered required rates of return—that is, the minimum rate of return an investor must receive in order to accept the investment. Second, interest rates can be considered discount rates. In the example above, 5.26 percent is that rate at which we discounted the $10,000 future amount to find its value today. Thus, we use the terms "interest rate" and "discount rate" almost interchangeably. Third, interest rates can be considered opportunity costs.

1 Examples in this reading and other readings in quantitative methods at Level I were updated in 2018 by Professor Sanjiv Sabherwal of the University of Texas, Arlington.

An **opportunity cost** is the value that investors forgo by choosing a particular course of action. In the example, if the party who supplied \$9,500 had instead decided to spend it today, he would have forgone earning 5.26 percent on the money. So we can view 5.26 percent as the opportunity cost of current consumption.

Economics tells us that interest rates are set in the marketplace by the forces of supply and demand, where investors are suppliers of funds and borrowers are demanders of funds. Taking the perspective of investors in analyzing market-determined interest rates, we can view an interest rate $r$ as being composed of a real risk-free interest rate plus a set of four premiums that are required returns or compensation for bearing distinct types of risk:

$$r = \text{Real risk-free interest rate} + \text{Inflation premium} + \text{Default risk premium} + \text{Liquidity premium} + \text{Maturity premium}$$

- The **real risk-free interest rate** is the single-period interest rate for a completely risk-free security if no inflation were expected. In economic theory, the real risk-free rate reflects the time preferences of individuals for current versus future real consumption.
- The **inflation premium** compensates investors for expected inflation and reflects the average inflation rate expected over the maturity of the debt. Inflation reduces the purchasing power of a unit of currency—the amount of goods and services one can buy with it. The sum of the real risk-free interest rate and the inflation premium is the **nominal risk-free interest rate**.[2] Many countries have governmental short-term debt whose interest rate can be considered to represent the nominal risk-free interest rate in that country. The interest rate on a 90-day US Treasury bill (T-bill), for example, represents the nominal risk-free interest rate over that time horizon.[3] US T-bills can be bought and sold in large quantities with minimal transaction costs and are backed by the full faith and credit of the US government.
- The **default risk premium** compensates investors for the possibility that the borrower will fail to make a promised payment at the contracted time and in the contracted amount.
- The **liquidity premium** compensates investors for the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly. US T-bills, for example, do not bear a liquidity premium because large amounts can be bought and sold without affecting their market price. Many bonds of small issuers, by contrast, trade infrequently after they are issued; the interest rate on such bonds includes a liquidity premium reflecting the relatively high costs (including the impact on price) of selling a position.
- The **maturity premium** compensates investors for the increased sensitivity of the market value of debt to a change in market interest rates as maturity is extended, in general (holding all else equal). The difference between the

2 Technically, 1 plus the nominal rate equals the product of 1 plus the real rate and 1 plus the inflation rate. As a quick approximation, however, the nominal rate is equal to the real rate plus an inflation premium. In this discussion we focus on approximate additive relationships to highlight the underlying concepts.

3 Other developed countries issue securities similar to US Treasury bills. The French government issues BTFs or negotiable fixed-rate discount Treasury bills (*Bons du Trésor àtaux fixe et à intérêts précomptés*) with maturities of up to one year. The Japanese government issues a short-term Treasury bill with maturities of 6 and 12 months. The German government issues at discount both Treasury financing paper (*Finanzierungsschätze des Bundes* or, for short, *Schätze*) and Treasury discount paper (*Bubills*) with maturities up to 24 months. In the United Kingdom, the British government issues gilt-edged Treasury bills with maturities ranging from 1 to 364 days. The Canadian government bond market is closely related to the US market; Canadian Treasury bills have maturities of 3, 6, and 12 months.

interest rate on longer-maturity, liquid Treasury debt and that on short-term Treasury debt reflects a positive maturity premium for the longer-term debt (and possibly different inflation premiums as well).

Using this insight into the economic meaning of interest rates, we now turn to a discussion of solving time value of money problems, starting with the future value of a single cash flow.

# 3 FUTURE VALUE OF A SINGLE CASH FLOW

- ☐ calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows
- ☐ demonstrate the use of a time line in modeling and solving time value of money problems

In this section, we introduce time value associated with a single cash flow or lump-sum investment. We describe the relationship between an initial investment or **present value (PV)**, which earns a rate of return (the interest rate per period) denoted as $r$, and its **future value (FV)**, which will be received $N$ years or periods from today.

The following example illustrates this concept. Suppose you invest $100 (PV = $100) in an interest-bearing bank account paying 5 percent annually. At the end of the first year, you will have the $100 plus the interest earned, 0.05 × $100 = $5, for a total of $105. To formalize this one-period example, we define the following terms:

PV = present value of the investment

$FV_N$ = future value of the investment $N$ periods from today

$r$ = rate of interest per period

For $N$ = 1, the expression for the future value of amount PV is

$$FV_1 = PV(1 + r) \tag{1}$$

For this example, we calculate the future value one year from today as $FV_1$ = $100(1.05) = $105.

Now suppose you decide to invest the initial $100 for two years with interest earned and credited to your account annually (annual compounding). At the end of the first year (the beginning of the second year), your account will have $105, which you will leave in the bank for another year. Thus, with a beginning amount of $105 (PV = $105), the amount at the end of the second year will be $105(1.05) = $110.25. Note that the $5.25 interest earned during the second year is 5 percent of the amount invested at the beginning of Year 2.

Another way to understand this example is to note that the amount invested at the beginning of Year 2 is composed of the original $100 that you invested plus the $5 interest earned during the first year. During the second year, the original principal again earns interest, as does the interest that was earned during Year 1. You can see how the original investment grows:

| | |
|---|---|
| Original investment | $100.00 |
| Interest for the first year ($100 × 0.05) | 5.00 |
| Interest for the second year based on original investment ($100 × 0.05) | 5.00 |

| | |
|---|---|
| Interest for the second year based on interest earned in the first year (0.05 × \$5.00 interest on interest) | 0.25 |
| Total | \$110.25 |

The \$5 interest that you earned each period on the \$100 original investment is known as **simple interest** (the interest rate times the principal). **Principal** is the amount of funds originally invested. During the two-year period, you earn \$10 of simple interest. The extra \$0.25 that you have at the end of Year 2 is the interest you earned on the Year 1 interest of \$5 that you reinvested.

The interest earned on interest provides the first glimpse of the phenomenon known as **compounding**. Although the interest earned on the initial investment is important, for a given interest rate it is fixed in size from period to period. The compounded interest earned on reinvested interest is a far more powerful force because, for a given interest rate, it grows in size each period. The importance of compounding increases with the magnitude of the interest rate. For example, \$100 invested today would be worth about \$13,150 after 100 years if compounded annually at 5 percent, but worth more than \$20 million if compounded annually over the same time period at a rate of 13 percent.

To verify the \$20 million figure, we need a general formula to handle compounding for any number of periods. The following general formula relates the present value of an initial investment to its future value after $N$ periods:

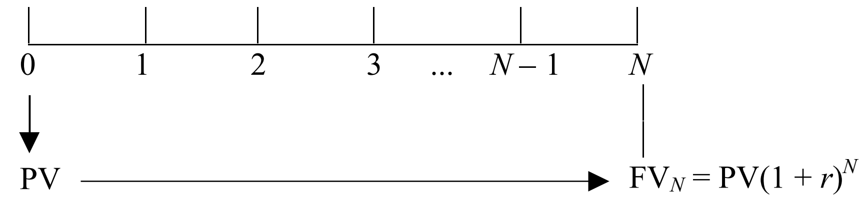
$$FV_N = PV(1 + r)^N \tag{2}$$

where $r$ is the stated interest rate per period and $N$ is the number of compounding periods. In the bank example, $FV_2 = \$100(1 + 0.05)^2 = \$110.25$. In the 13 percent investment example, $FV_{100} = \$100(1.13)^{100} = \$20,316,287.42$.

The most important point to remember about using the future value equation is that the stated interest rate, $r$, and the number of compounding periods, $N$, must be compatible. Both variables must be defined in the same time units. For example, if $N$ is stated in months, then $r$ should be the one-month interest rate, unannualized.

A time line helps us to keep track of the compatibility of time units and the interest rate per time period. In the time line, we use the time index $t$ to represent a point in time a stated number of periods from today. Thus the present value is the amount available for investment today, indexed as $t = 0$. We can now refer to a time $N$ periods from today as $t = N$. The time line in Exhibit 1 shows this relationship.

**Exhibit 1: The Relationship between an Initial Investment, PV, and Its Future Value, FV**




In Exhibit 1, we have positioned the initial investment, PV, at $t = 0$. Using Equation 2, we move the present value, PV, forward to $t = N$ by the factor $(1 + r)^N$. This factor is called a future value factor. We denote the future value on the time line as FV and

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position it at $t = N$. Suppose the future value is to be received exactly 10 periods from today's date ($N = 10$). The present value, PV, and the future value, FV, are separated in time through the factor $(1 + r)^{10}$.

The fact that the present value and the future value are separated in time has important consequences:

- We can add amounts of money only if they are indexed at the same point in time.
- For a given interest rate, the future value increases with the number of periods.
- For a given number of periods, the future value increases with the interest rate.

To better understand these concepts, consider three examples that illustrate how to apply the future value formula.

**EXAMPLE 1**

### The Future Value of a Lump Sum with Interim Cash Reinvested at the Same Rate

1. You are the lucky winner of your state's lottery of $5 million after taxes. You invest your winnings in a five-year certificate of deposit (CD) at a local financial institution. The CD promises to pay 7 percent per year compounded annually. This institution also lets you reinvest the interest at that rate for the duration of the CD. How much will you have at the end of five years if your money remains invested at 7 percent for five years with no withdrawals?

### Solution:

To solve this problem, compute the future value of the $5 million investment using the following values in Equation 2:

$$
\begin{aligned}
\text{PV} &= \$5{,}000{,}000 \\
r &= 7\% = 0.07 \\
N &= 5 \\
\text{FV}_N &= \text{PV}(1 + r)^N \\
&= \$5{,}000{,}000(1.07)^5 \\
&= \$5{,}000{,}000(1.402552) \\
&= \$7{,}012{,}758.65
\end{aligned}
$$

At the end of five years, you will have $7,012,758.65 if your money remains invested at 7 percent with no withdrawals.

*In this and most examples in this reading, note that the factors are reported at six decimal places but the calculations may actually reflect greater precision.* For example, the reported 1.402552 has been rounded up from 1.40255173 (the calculation is actually carried out with more than eight decimal places of precision by the calculator or spreadsheet). Our final result reflects the higher number of decimal places carried by the calculator or spreadsheet.[4]

4 We could also solve time value of money problems using tables of interest rate factors. Solutions using tabled values of interest rate factors are generally less accurate than solutions obtained using calculators or spreadsheets, so practitioners prefer calculators or spreadsheets.

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**EXAMPLE 2**

## The Future Value of a Lump Sum with No Interim Cash

1. An institution offers you the following terms for a contract: For an investment of ¥2,500,000, the institution promises to pay you a lump sum six years from now at an 8 percent annual interest rate. What future amount can you expect?

### Solution:

Use the following data in Equation 2 to find the future value:

$$\begin{aligned} \text{PV} &= ¥2,500,000 \\ r &= 8\% = 0.08 \\ N &= 6 \\ \text{FV}_N &= \text{PV}(1+r)^N \\ &= ¥2,500,000(1.08)^6 \\ &= ¥2,500,000(1.586874) \\ &= ¥3,967,186 \end{aligned}$$

You can expect to receive ¥3,967,186 six years from now.

Our third example is a more complicated future value problem that illustrates the importance of keeping track of actual calendar time.

**EXAMPLE 3**

## The Future Value of a Lump Sum

1. A pension fund manager estimates that his corporate sponsor will make a $10 million contribution five years from now. The rate of return on plan assets has been estimated at 9 percent per year. The pension fund manager wants to calculate the future value of this contribution 15 years from now, which is the date at which the funds will be distributed to retirees. What is that future value?

### Solution:

By positioning the initial investment, PV, at $t = 5$, we can calculate the future value of the contribution using the following data in Equation 2:

$$\begin{aligned} \text{PV} &= \$10 \text{ million} \\ r &= 9\% = 0.09 \\ N &= 10 \\ \text{FV}_N &= \text{PV}(1+r)^N \\ &= \$10,000,000(1.09)^{10} \\ &= \$10,000,000(2.367364) \\ &= \$23,673,636.75 \end{aligned}$$

This problem looks much like the previous two, but it differs in one important respect: its timing. From the standpoint of today ($t = 0$), the future amount of $23,673,636.75 is 15 years into the future. Although the future value is 10 years from its present value, the present value of $10 million will not be received for another five years.

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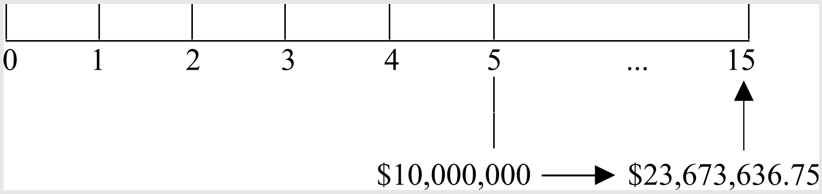
**Exhibit 2: The Future Value of a Lump Sum, Initial Investment Not at $t = 0$**




As Exhibit 2 shows, we have followed the convention of indexing today as $t = 0$ and indexing subsequent times by adding 1 for each period. The additional contribution of $10 million is to be received in five years, so it is indexed as $t = 5$ and appears as such in the figure. The future value of the investment in 10 years is then indexed at $t = 15$; that is, 10 years following the receipt of the $10 million contribution at $t = 5$. Time lines like this one can be extremely useful when dealing with more-complicated problems, especially those involving more than one cash flow.

In a later section of this reading, we will discuss how to calculate the value today of the $10 million to be received five years from now. For the moment, we can use Equation 2. Suppose the pension fund manager in Example 3 above were to receive $6,499,313.86 today from the corporate sponsor. How much will that sum be worth at the end of five years? How much will it be worth at the end of 15 years?

$$\begin{aligned}
\text{PV} &= \$6{,}499{,}313.86 \\
r &= 9\% = 0.09 \\
N &= 5 \\
\text{FV}_N &= \text{PV}(1 + r)^N \\
&= \$6{,}499{,}313.86(1.09)^5 \\
&= \$6{,}499{,}313.86(1.538624) \\
&= \$10{,}000{,}000 \text{ at the five-year mark}
\end{aligned}$$

and

$$\begin{aligned}
\text{PV} &= \$6{,}499{,}313.86 \\
r &= 9\% = 0.09 \\
N &= 15 \\
\text{FV}_N &= \text{PV}(1 + r)^N \\
&= \$6{,}499{,}313.86(1.09)^{15} \\
&= \$6{,}499{,}313.86(3.642482) \\
&= \$23{,}673{,}636.74 \text{ at the 15-year mark}
\end{aligned}$$

These results show that today's present value of about $6.5 million becomes $10 million after five years and $23.67 million after 15 years.

# 4 NON-ANNUAL COMPOUNDING (FUTURE VALUE)

☐ calculate the solution for time value of money problems with different frequencies of compounding